Homework IV
Due Date: 04/04/2022

Exercise 1 (2 points). We consider a C^2 real solution u of the wave equation

$$\partial_t^2 u - \partial_x^2 u = 0,$$

in the cylinder $\mathcal{C} = \{(t,x) : t > 0, a \leq x \leq b\} \subset \mathbb{R}_t \times \mathbb{R}_x$. Assume that u satisfies the boundary conditions

$$u(t,a) = (\partial_t u + \partial_x u)(t,b) = 0.$$

Show that for $t \geq 2(b-a)$, $u \equiv 0$.

Exercise 2 (3 points). (a) Assume $\mathbf{E} = (E^1, E^2, E^3)$ and $\mathbf{B} = (B^1, B^2, B^3)$ solve Maxwell's equations

$$\begin{cases} \partial_t \mathbf{E} = \text{curl } \mathbf{B}, \quad \partial_t \mathbf{B} = -\text{curl } \mathbf{E}, \\ \text{div } \mathbf{B} = \text{div } \mathbf{E} = 0. \end{cases}$$

Show that

$$\partial_t^2 \mathbf{E} - \Delta \mathbf{E} = \partial_t^2 \mathbf{B} - \Delta \mathbf{B} = \mathbf{0}.$$

(b). Assume that $\mathbf{u} = (u^1, u^2, u^3)$ solves the evolution equations of linear elasticity

$$\partial_t^2 \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) D(\text{div } \mathbf{u}) = 0, \quad \text{in } (0,\infty) \times \mathbb{R}^3.$$

Show that $w := \text{div } \mathbf{u}$ and $\mathbf{v} := \text{curl } \mathbf{u}$ each solve wave equations, but with differing speeds of propagation.

Exercise 3 (3 points) Let u solve the initial-value problem for the 1D wave equation:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, \quad \text{in } (0,\infty) \times \mathbb{R}, \\ (u, \partial_t u)_{|t=0} = (g,h), \quad \text{on } \mathbb{R}. \end{cases}$$

Suppose g, h have compact support. The kinetic energy is $k(t) := \frac{1}{2}\int_{\mathbb{R}} (\partial_t u(t,x))^2 \mathrm{d}x$ and the potential energy is $p(t) = \frac{1}{2}\int_{\mathbb{R}} \left(\partial_x u(t,x)\right)^2 \mathrm{d}x$. Prove

(a) $k(t) + p(t)$ is constant in t.
(b) $k(t) = p(t)$ for all large enough times t.

Exercise 4 (2 points) Consider the linear damping Klein-Gordon equation,

$$\partial_t^2 u + \partial_t u - \partial_x^2 u + u = 0.$$

We define the following energy for solution

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} \left((\partial_t u)^2 + (\partial_x u)^2 + u^2\right) \mathrm{d}x.$$

Show that

$$\lim_{t \to \infty} E(t) = 0.$$